

# Projective Expected Utility

Pierfrancesco La Mura

Department of Microeconomics and Information Systems  
HHL - Leipzig Graduate School of Management  
Jahnallee 59, 04109 Leipzig (Germany)

## Abstract

Motivated by several classic decision-theoretic paradoxes, and by analogies with the paradoxes which in physics motivated the development of quantum mechanics, we introduce a projective generalization of expected utility along the lines of the quantum-mechanical generalization of probability theory. The resulting decision theory accommodates the dominant paradoxes, while retaining significant simplicity and tractability. In particular, every finite game within this larger class of preferences still has an equilibrium.

## Introduction

John von Neumann (1903-1957) is widely regarded as a founding father of many fields, including game theory, decision theory, quantum mechanics and computer science. Two of his contributions are of special importance in our context. In 1932, von Neumann proposed the first rigorous foundation for quantum mechanics, based on a calculus of projections in Hilbert spaces. In 1944, together with Oskar Morgenstern, he gave the first axiomatic foundation for the expected utility hypothesis (Bernoulli 1738). In both cases, the frameworks he pioneered are still at the core of the respective fields. In particular, the expected utility hypothesis is still the *de facto* foundation of fields such as finance and game theory.

The von Neumann - Morgenstern axiomatization of expected utility, and later on the subjective formulations by Savage (1954) and Anscombe and Aumann (1963) were immediately greeted as simple and intuitively compelling. Yet, in the course of time, a number of empirical violations and paradoxes (Allais 1953, Ellsberg 1961, Rabin and Thaler 2001) came to cast doubt on the validity of the hypothesis as a foundation for the theory of rational decisions in conditions of risk and subjective uncertainty. In economics and in the social sciences, the shortcomings of the expected utility hypothesis are generally well known, but often tacitly accepted in view of the great tractability and usefulness of the corresponding mathematical framework. In fact, the hypothesis postulates that preferences can be represented by way of a utility functional which is linear in probabilities, and linearity makes expected utility representations particularly tractable in models and applications.

The experimental paradoxes, which in the context of physics motivated the introduction of quantum mechanics, bear interesting relations with some of the decision-theoretic paradoxes which came to challenge the status of expected utility in the social sciences: in both cases, the anomalies can be regarded as violations of an appropriately defined notion of independence. In physics, quantum mechanics was introduced as a tractable and empirically accurate mathematical framework in the presence of such violations. In economics and the social sciences, the importance of accounting for violations of the expected utility hypothesis has long been recognized, but so far none of its numerous alternatives has emerged as dominant, sometimes due to a lack of mathematical tractability, or to the ad-hoc nature of some axiomatic proposals. Motivated by these considerations, we would like to introduce a decision-theoretic framework which accommodates the dominant paradoxes while retaining significant simplicity and tractability. As we shall see, this is obtained by weakening the expected utility hypothesis to its projective counterpart, in analogy with the quantum-mechanical generalization of classical probability theory.

The structure of the paper is as follows. The next section briefly reviews the von Neumann-Morgenstern framework. Sections 3 and 4, respectively, present Allais' and Ellsberg's paradoxes. Section 5 introduces a mathematical framework for projective expected utility, and a representation result. Section 6 contains a subjective formulation for projective expected utility, and a corresponding representation result. In section 7 there is a brief discussion, and in sections 8 and 9, respectively, we show how Allais' and Ellsberg's paradoxes can be accommodated within the new framework. Section 10 discusses the multi-agent case and the issue of existence of strategic equilibrium, while the last section concludes.

## von Neumann - Morgenstern Expected Utility

Let  $S$  be a finite set of outcomes, and  $\Delta$  be the set of probability functions defined on  $S$ , taken to represent risky prospects (or *lotteries*). Next, let  $\succeq$  be a complete and transitive binary relation defined on  $\Delta \times \Delta$ , representing a decision-maker's preference ordering over lotteries. Indifference of  $p, q \in \Delta$  is defined as  $[p \succeq q \text{ and } q \succeq p]$  and denoted as  $p \sim q$ , while strict preference of  $p$  over  $q$  is defined as  $[p \succeq q \text{ and not } q \succeq p]$ , and denoted by  $p \succ q$ . The preference ordering is assumed to satisfy the following two

conditions.

**Axiom 1** (Archimedean) For all  $p, q, r \in \Delta$  with  $p \succ q \succ r$ , there exist  $\alpha, \beta \in (0, 1)$  such that  $\alpha p + (1 - \alpha)r \succ q \succ \beta p + (1 - \beta)r$ .

**Axiom 2** (Independence) For all  $p, q, r \in \Delta$ ,  $p \succeq q$  if, and only if,  $\alpha p + (1 - \alpha)r \succeq \alpha q + (1 - \alpha)r$  for all  $\alpha \in [0, 1]$ .

A functional  $u : \Delta \rightarrow R$  is said to represent  $\succeq$  if, for all  $p, q \in \Delta$ ,  $p \succeq q$  if and only if  $u(p) \geq u(q)$ .

**Theorem 1** (von Neumann and Morgenstern) Axioms 1 and 2 are jointly equivalent to the existence of a functional  $u : \Delta \rightarrow R$  which represents  $\succeq$  such that, for all  $p \in \Delta$ ,

$$u(p) = \sum_{s \in S} u(s)p(s).$$

The von Neumann - Morgenstern setting is appropriate whenever the nature of the uncertainty is purely objective: all lotteries are associated to objective random devices, such as dice or roulette wheels, with well-defined and known frequencies for all outcomes, and the decision-maker only evaluates a lottery based on the frequencies of its outcomes.

## The “Double-Slit Paradox” of Decision Theory

The following paradox is due to Allais (1953). First, please choose between:

A: A chance of winning 4000 dollars with probability 0.2 (expected value 800 dollars)

B: A chance of winning 3000 dollars with probability 0.25 (expected value 750 dollars).

Next, please choose between:

C: A chance of winning 4000 dollars with probability 0.8 (expected value 3200 dollars)

B: A chance of winning 3000 dollars with certainty.

If you chose  $A$  over  $B$ , and  $D$  over  $C$ , then you are in the modal class of respondents. The paradox lies in the observation that  $A$  and  $C$  are special cases of a two-stage lottery  $E$  which in the first stage either returns zero dollars with probability  $(1 - \alpha)$  or, with probability  $\alpha$ , it leads to a second stage where one gets 4000 dollars with probability 0.8 and zero otherwise. In particular, if  $\alpha$  is set to 1 then  $E$  reduces to  $C$ , and if  $\alpha$  is set to 0.25 it reduces to  $A$ . Similarly,  $B$  and  $D$  are special cases of a two-stage lottery  $F$  which again with probability  $(1 - \alpha)$  returns zero, and with probability  $\alpha$  continues to a second stage where one wins 3000 dollars with probability 1. Again, if  $\alpha = 1$  then  $F$  reduces to  $D$ , and if  $\alpha = 0.25$  it reduces to  $B$ . Then it is easy to see that the  $[A \succ B, D \succ C]$  pattern violates Axiom 2 (Independence), as  $E$  can be regarded as a lottery  $\alpha p + (1 - \alpha)r$ , and  $F$  as a lottery  $\alpha q + (1 - \alpha)r$ , where  $p$  and  $q$  represent lottery  $C$  and  $D$  respectively and  $r$  represents the lottery in which one gets zero dollars with certainty. When comparing  $E$  and  $F$ , why should it matter what is the value of  $\alpha$ ? Yet, experimentally one finds that it does.

Allais' paradox bears a certain resemblance with the well-known double-slit paradox in physics. Imagine cutting two parallel slits in a sheet of cardboard, shining light through them, and observing the resulting pattern as particles going through the two slits scatter on a wall behind the cardboard barrier. Experimentally, one finds that when both slits are open, the overall scattering pattern is not the sum of the two scattering patterns produced when one slit is open and the other closed. The effect does not go away even if particles of light are shone one by one, and this is paradoxical: why should it matter to an individual particle which happens to go through the left slit, when determining where to scatter later on, with what probability it could have gone through the right slit instead? In a sense, each particle in the double-slit experiment behaves like a decision-maker who violates the Independence axiom in Allais' experiment.

## Ellsberg's Paradox

Another disturbing violation of the expected utility hypothesis was pointed out by Ellsberg (1961). Suppose that a box contains 300 balls of three possible colors: red ( $R$ ), green ( $G$ ), and blue ( $B$ ). You know that the box contains exactly 100 red balls, but are given no information on the proportion of green and blue.

You win if you guess which color will be drawn. Do you prefer to bet on red ( $R$ ) or on green ( $G$ )? Many respondents choose  $R$ , on grounds that the probability of drawing a red ball is known to be  $1/3$ , while the only information on the probability of drawing a green ball is that it is between 0 and  $2/3$ . Now suppose that you win if you guess which color will *not* be drawn. Do you prefer to bet that red will not be drawn ( $\bar{R}$ ) or that green will not be drawn ( $\bar{G}$ )? Again many respondents prefer to bet on  $\bar{R}$ , as the probability is known ( $2/3$ ) while the probability of  $\bar{G}$  is only known to be between 1 and  $1/3$ .

The pattern  $[R \succ G, \bar{R} \succ \bar{G}]$  is incompatible with von Neumann - Morgenstern expected utility, which only deals with known probabilities, and is also incompatible with the Savage (1954) formulation of expected utility with subjective probability as it violates its Sure Thing axiom. The paradox suggests that, in order to account for such patterns of choice, subjective uncertainty and risk should be handled as distinct notions. Once again, the issue bears interesting relations with quantum mechanics, where the need to keep a distinction between subject-related uncertainty and objective risk also presented itself, and is naturally met in the formalism by the distinction between pure and mixed states.

## Projective Expected Utility

Let  $X$  be the positive orthant of the unit sphere in  $\mathbb{R}^n$ , where  $n$  is the cardinality of the set of relevant outcomes  $S := \{s_1, s_2, \dots, s_n\}$ . Then von Neumann - Morgenstern lotteries, regarded as elements of the unit simplex, are in one-to-one correspondence with elements of  $X$ , which can therefore be interpreted as risky prospects, for which the frequencies of the relevant outcomes are fully known. Observe that, while the projections of elements of the unit simplex (and hence,  $L^1$  unit vectors) on the basis vectors can be nat-

urally associated with probabilities, if we choose to model von Neumann - Morgenstern lotteries as elements of the unit sphere (and hence, as unit vectors in  $L^2$ ) then probabilities are naturally associated with *squared* projections. The advantage of such move is that  $L^2$  is the only  $L^p$  space which is also a Hilbert space, and Hilbert spaces have a very tractable projective structure which is exploited by the representation. In particular, it is unique to  $L^2$  that the set of unit vectors is invariant with respect to projections.

Next, let  $\langle \cdot | \cdot \rangle$  denote the usual inner product in  $\mathbb{R}^n$ . An orthonormal basis is a set of unit vectors  $(b_1, \dots, b_n)$  such that  $\langle b_i | b_j \rangle = 0$  whenever  $i \neq j$ . The natural basis corresponds to the set of degenerate lotteries returning each objective lottery outcome with certainty, and is conveniently identified with the set of objective lottery outcomes. Yet, in any realistic experimental setting, it is very unlikely that those objective outcomes will happen to coincide with the relevant outcomes as subjectively perceived by the decision-maker. Moreover, even if those could be fully elicited, it would be generally problematic to relate a von Neumann-Morgenstern lottery, which only specifies the probabilities of the objective outcomes, with the decision-relevant probabilities induced on the subjective outcomes. The perspective of the observer or modeller is inesorably bound to objectively measurable entities, such as frequencies and prizes; by contrast, the decision-maker thinks and acts based on subjective preferences and subjective outcomes, which in a revealed-preference context should be presumed to exist while at the same time assumed, as a methodological principle, to be inaccessible to direct measurement.

**Axiom 3 (Born's Rule)** *There exists an orthonormal basis  $(z_1, \dots, z_n)$  such that, for all  $x \in X$  and all  $s_i \in S$ , any two lotteries are indifferent whenever their risk profiles*

$$p_x(z_i) = \langle x | z_i \rangle^2, i = 1, \dots, n$$

*coincide.*

Axiom 3 requires that there exist an interpretation of elements of  $X$  in terms of the relevant dimensions of risk as perceived by the decision-maker. In the von Neumann - Morgenstern treatment, Axiom 3 is tacitly assumed to hold with respect to the natural basis in  $\mathbb{R}^n$ . This implicit assumption amounts to the requirement that lotteries are only evaluated based on the probabilities they induce on the objective lottery outcomes.

By contrast, the *preferred basis* postulated in Axiom 3 is allowed to vary across different decision-makers, capturing the idea that the relevant dimensions of risk (that is, those pertaining to the actual subjective consequences) may be perceived differently by different subjects, perhaps due to portfolio effects, while the natural basis may be taken to represent the relevant dimensions of risk as perceived by the modeler or an external observer (that is, in terms of the objective outcomes as evaluated by the observer). Portfolio effects are very difficult to exclude in an experimental setting. For instance, the very fact of proposing to a subject the Allais or Ellsberg games described above generates an expectation of gain. An invitation to play the game can be effectively regarded as a risky security which is donated

to the subject, and whose subjective returns are obviously correlated, but do not necessarily coincide, with the monetary outcomes of the experiment. Even in such simple contexts, significant hedging behavior cannot be in principle excluded.

In our context, orthogonality captures the idea that two events or outcomes are mutually exclusive (for one event to have probability one, the other must have probability zero). The preferred basis captures which, among all possible ways to partition the relevant uncertainty into a set of mutually exclusive events or outcomes, leads to a set of payoff-relevant orthogonal lotteries from which preferences on all other lotteries can be assigned in a linear fashion. In particular, the basis elements must span the whole range of preferences. For instance, in case a decision-maker is indifferent between two outcomes, but strictly prefers to receive both of them with equal frequency, preferences on the two outcomes do not span all the relevant range; therefore, this preference pattern cannot be captured in the natural basis, and hence in von Neumann - Morgenstern expected utility. Axiom 3 postulates that each lottery is evaluated by the decision maker solely by the uncertainty it induces on appropriately chosen payoff-relevant dimensions. For now, and only for simplicity, we assume that the cardinality of the set of subjective outcomes coincides with that of the objective ones; we relax this assumption later on, in the subjective formulation.

Once an orthonormal basis  $Z$  is given, each objective lottery  $x$  uniquely corresponds to a function  $p_x : Z \rightarrow [0, 1]$ , such that  $p_x(z_i) = \langle x | z_i \rangle^2$  for all  $z_i \in Z$ . Let  $B$  be the set of all such *risk profiles*  $p_x$ , for  $x \in X$ , and let  $\succeq$  be the complete and transitive preference ordering induced on  $B \times B$  by preferences on the underlying lotteries. We postulate the following two axioms, which mirror those in the von Neumann - Morgenstern treatment.

**Axiom 4 (Archimedean)** *For all  $x, y, z \in X$  with  $p(x) \succ p(y) \succ p(z)$ , there exist  $\alpha, \beta \in (0, 1)$  such that  $\alpha p(x) + (1 - \alpha)p(z) \succ p(y) \succ \beta p(x) + (1 - \beta)p(z)$ .*

**Axiom 5 (Independence)** *For all  $x, y, z \in X$ ,  $p_x \succeq p_y$  if, and only if,  $\alpha p_x + (1 - \alpha)p_z \succeq \alpha p_y + (1 - \alpha)p_z$  for all  $\alpha \in [0, 1]$ .*

Observe that the two axioms impose conditions solely on risk profiles, and not on the underlying lotteries. In particular, note that a convex combination  $\alpha p_x + (1 - \alpha)p_y$ , where  $p_x$  and  $p_y$  are objective probability functions, is also an objective probability function. Hence, the type of mixing in Axioms 4 and 5 can be regarded as objective, while subjective mixing will be later on captured by convex combinations (in  $\mathbb{R}^n$ ) of the underlying (Anscombe-Aumann) acts.

**Theorem 2** *Axioms 3-5 are jointly equivalent to the existence of a symmetric matrix  $U$  such that  $u(x) := x'Ux$  for all  $x \in X$  represents  $\succeq$ .*

**Proof.** Assume that Axiom 3 holds with respect to a given orthonormal basis  $(z_1, \dots, z_n)$ . By the von Neumann - Morgenstern result (which applies to any convex mixture set,

such as B) Axioms 4 and 5 are jointly equivalent to the existence of a functional  $u$  which represents the ordering and is linear in  $p$ , i.e.

$$u(x) = \sum_{i=1}^n u(s_i) p_{s_i}(x) = \sum_{i=1}^n u(s_i) \langle x | z_i \rangle^2,$$

where the second equality is by definition of  $p$  as squared inner product with respect to the preferred basis. The above can be equivalently written, in matrix form, as

$$u(x) = x' P' D P x = x' U x,$$

where  $D$  is the diagonal matrix with the payoffs on the main diagonal,  $P$  is the projection matrix associated to  $(z_1, \dots, z_n)$ , and  $U := P' D P$  is symmetric. Conversely, by the Spectral Decomposition theorem, for any symmetric matrix  $U$  there exist a diagonal matrix  $D$  and a projection matrix  $P$  such that  $U = P' D P$ , and hence

$$x' U x = x' P' D P x$$

for all  $x \in X$ . But this is just expected utility with respect to the orthonormal basis defined by  $P$ . Hence, the three axioms are jointly equivalent to the existence of a symmetric matrix  $U$  such that  $u(x) := x' U x$  represents the preference ordering. QED

## Subjective Formulation

We introduce the following setup and notation.

$\mathcal{S}$  is a finite set of states of Nature.

$\langle \cdot | \cdot \rangle$  denotes the usual inner product in Euclidean space.

$\Omega$  is the natural basis in  $\mathbb{R}^n$ , identified with a finite set  $\{\omega^1, \dots, \omega^n\}$  of lottery outcomes (prizes).

$Z$  is an orthonormal basis in  $\mathbb{R}^m$ , with  $m \geq n$ , identified with a finite set of subjective outcomes or consequences  $\{z^1, \dots, z^m\}$ .  $V$  is an arbitrary  $(m \times n)$  matrix chosen so that, for all  $\omega^i$  in  $\Omega$ ,  $V \omega^i$  is a unit vector in  $\mathbb{R}^m$ .

The quantity  $\langle z^j | V \omega^i \rangle^2$  is interpreted as  $p(z^j | \omega^i)$ , the conditional probability of subjective outcome  $z^j$  given objective outcome  $\omega^i$ .

Lotteries correspond to  $L^2$  unit vectors  $x \in \mathbb{R}_+^n$ ;  $X$  is the set of all lotteries.

Since  $\Omega$  is the natural basis,  $\langle \omega^i | x \rangle^2 = x_i^2$ ; this quantity is interpreted as  $p(\omega^i | x)$ .

Once the subjective consequences  $z^j$  are specified, for any lottery  $x$  one can readily compute  $p(\omega^i | x) = x_i^2$  and  $p(z^j | x) = \sum_i p(\omega^i | x) p(z^j | \omega^i) = \sum_i x_i^2 \langle z^j | V \omega^i \rangle^2$ . Moreover, given the latter probabilistic constraints, one can readily identify the unique lottery  $x$  and an orthonormal basis  $Z$  which jointly satisfy them. Hence, in the above construction lotteries are identified with respect to two different frames of reference: objective lottery outcomes, and subjective consequences.

An act is identified with a function  $f : \mathcal{S} \rightarrow X$ .  $H$  is the set of all acts.

$\Delta(X)$  is the (nonempty, closed and convex) set of all probability functions on  $Z$  induced by lotteries in  $X$ .

$M$  is the set of all vectors  $(p_s)_{s \in \mathcal{S}}$ , with  $p_s \in \Delta(X)$ .

For each  $f \in H$  a corresponding *risk profile*  $p^f \in M$  is defined, for all  $s \in \mathcal{S}$  and all  $z^j \in Z$ , by  $p_s^f(z^j) := \langle z^j | V f_s \rangle^2$ . Observe that, even though  $p_s^f$  is a well-defined probability function,  $p_s^f(z^j)$  generally differs from the probability of  $z^j$  given  $f(s)$ , which is given by  $\sum_i f_i(s)^2 \langle z^j | V \omega^i \rangle^2$ . Hence, except when the two measures happen to coincide, the risk profile  $p^f$  cannot be interpreted as a vector of objective probabilities, but rather as a possible sufficient statistic for the ranking of acts from the point of view of the decision-maker.

As customary, we assume that the decision-makers preferences are characterized by a rational (i.e., complete and transitive) preference ordering  $\succeq$  on acts.

Next, we proceed with the following assumptions, which mirror those in Anscombe and Aumann (1963).

**Axiom 6 (Projective)** *There exists an orthonormal basis  $Z$  such that any two acts  $f, g \in H$  are indifferent if  $p^f = p^g$ .*

In Anscombe and Aumann's setting, the above axiom is implicitly assumed to hold with  $Z \equiv \Omega$  and  $V \equiv I$ , where  $I$  is the  $(n \times n)$  identity matrix. Because of Axiom 6, preferences on acts can be equivalently expressed as preferences on risk profiles. For all  $p^f, p^g \in M$ , we stipulate that  $p^f \succeq p^g$  if and only if  $f \succeq g$ .

**Axiom 7 (Archimedean)** *If  $p^f, p^g, p^h \in M$  are such that  $p^f \succ p^g \succ p^h$ , then there exist  $a, b \in (0, 1)$  such that  $ap^f + (1-a)p^h \succ p^g \succ bp^f + (1-b)p^h$ .*

**Axiom 8 (Independence)** *For all  $p^f, p^g, p^h \in M$ , and for all  $a \in (0, 1]$ ,  $p^f \succ p^g$  if and only if  $ap^f + (1-a)p^h \succ ap^g + (1-a)p^h$ .*

**Axiom 9 (Non-degeneracy)** *There exist  $p^f, p^g \in M$  such that  $p^f \succ p^g$ .*

**Axiom 10 (State independence)** *Let  $s, t \in \mathcal{S}$  be non-null states, and let  $p, q \in \Delta(X)$ . Then, for any  $p^f \in M$ ,*

$$\begin{aligned} (p_1^f, \dots, p_{s-1}^f, p, p_{s+1}^f, \dots, p_n^f) &\succ \\ (p_1^f, \dots, p_{s-1}^f, q, p_{s+1}^f, \dots, p_n^f) & \\ \text{if, and only if,} & \\ (p_1^f, \dots, p_{t-1}^f, p, p_{t+1}^f, \dots, p_n^f) &\succ \\ (p_1^f, \dots, p_{t-1}^f, q, p_{t+1}^f, \dots, p_n^f) &. \end{aligned}$$

**Theorem 3 (Anscombe and Aumann)** *The preference relation  $\succeq$  fulfills Axioms 6 – 10 if and only if there is a unique probability measure  $\pi$  on  $\mathcal{S}$  and a non-constant function  $u : X \rightarrow \mathbb{R}$  (unique up to positive affine rescaling) such that, for any  $f, g \in H$ ,  $f \succeq g$  if and only if*

$$\sum_{s \in \mathcal{S}} \pi(s) \sum_{z^i \in Z} p_s^f(z^i) u(z^i) \geq \sum_{s \in \mathcal{S}} \pi(s) \sum_{z^i \in Z} p_s^g(z^i) u(z^i).$$

**Theorem 4** *The preference relation  $\succeq$  fulfills Axioms 6 – 10 if and only if there is a unique probability measure  $\pi$  on  $\mathcal{S}$  and a symmetric  $(n \times n)$  matrix  $U$  with non-constant eigenvalues such that, for any  $f, g \in H$ ,  $f \succeq g$  if and only if  $\sum_{s \in \mathcal{S}} \pi(s) f_s' U f_s \geq \sum_{s \in \mathcal{S}} \pi(s) g_s' U g_s$ .*

**Proof.** Let  $D$  be the  $(m \times m)$  diagonal matrix defined by  $D_{i,i} = u(z^i)$ , and let  $P$  be the projection matrix defined by  $(P_{i,\cdot})' = z^i$ . If Axioms 6-10 hold, we know from Theorem 3 that the preference ordering has an expected utility representation. Observe that, since  $(PVf_s)_i^2 = \langle z^i | Vf_s \rangle^2 = p_s^f(z^i)$ ,  $\sum_{s \in \mathcal{S}} \pi(s) \sum_{z^i \in Z} p_s^f(z^i) u(z^i) = \sum_{s \in \mathcal{S}} \pi(s) f_s' V' P' D P V f_s$ .

It follows that the expected utility of any act  $f$  can be written as  $\sum_{s \in \mathcal{S}} \pi(s) f_s' U f_s$ , where  $U := V' P' D P V$  is a  $(n \times n)$  symmetric matrix. Since the diagonal elements of  $D$  are the eigenvalues of  $U$ , the latter matrix has non-constant eigenvalues.

Conversely, let  $U$  be a symmetric  $(n \times n)$  matrix with non-constant eigenvalues. By the Spectral Decomposition theorem there exist a projection matrix  $P$  and a diagonal matrix  $D$  such that  $U = V' P' D P V$ , and therefore  $\sum_{s \in \mathcal{S}} \pi(s) f_s' U f_s = \sum_{s \in \mathcal{S}} \pi(s) f_s' V' P' D P V f_s$ . Observe that the right-hand side is the expected utility of  $f$  with respect to the orthonormal basis  $Z$  defined by  $z^i = P'_{i,\cdot}$ , and with  $u(z^i) = D_{i,i}$ . Since the diagonal elements of  $D$  are the eigenvalues of  $U$ , if the latter has non-constant eigenvalues then  $u(z^i)$  is also non-constant, and hence by Theorem 3 Axioms 6-10 must hold. QED

The above formulation extends the representation to situations of subjective uncertainty. Say that an act is *pure* or *certain* if it returns for sure a given objective lottery, and *mixed* or *uncertain* if it can be obtained as the convex combination  $\alpha f + (1 - \alpha)g$  of two non-identical acts  $f$  and  $g$ .

### Properties of the Representation

Our representation generalizes the Anscombe-Aumann expected utility framework in three directions. First, subjective uncertainty (from mixed acts) and risk (from pure acts) are treated as distinct notions. Second, as we shall see, within this class of preferences both Allais' and Ellsberg's paradoxes are accommodated. Finally, the construction easily extends to the complex unit sphere, provided that  $\langle x|y \rangle^2$  is replaced by  $|\langle x|y \rangle|^2$  in the definition of  $p$ , in which case Theorems 2 and 4 hold with respect to a Hermitian (rather than symmetric) payoff matrix  $U$ , and the result also provides axiomatic foundations for decisions involving quantum uncertainty.

Let  $e_i, e_j$  be the degenerate lotteries assigning probability 1 to outcomes  $s_i$  and  $s_j$ , respectively, and let  $e_{i,j}$  be the lottery assigning probability 1/2 to each of the two states. Observe that, for any two distinct  $s_i$  and  $s_j$ ,

$$U_{ij} = u(e_{i,j}) - \left( \frac{1}{2}u(e_i) + \frac{1}{2}u(e_j) \right).$$

It follows that the off-diagonal entry  $U_{ij}$  in the payoff matrix can be interpreted as the discount, or premium, attached to an equiprobable combination of the two outcomes with respect to its expected utility base-line, and hence as a measure of preference for risk versus uncertainty along the specific dimension involving outcomes  $s_i$  and  $s_j$ .

Observe that the functional in Theorem 2 is quadratic in  $x$ , but linear in  $p$ . If  $U$  is diagonal, then its eigenvalues coincide with the diagonal elements. In von Neumann - Morgenstern expected utility, those eigenvalues contain all the relevant information about the decision-maker's risk attitudes. In our framework, risk attitudes are jointly captured by both the diagonal and non-diagonal elements of  $U$ , which are completely characterized by a diagonal matrix  $D$  with the eigenvalues of  $U$  on the main diagonal and a projection matrix  $P$ . Furthermore, in our setting attitudes towards uncertainty are captured by the concavity or convexity (in  $x$ ) of the quadratic form  $x' U x$ , and therefore, ultimately, by the definiteness condition of  $U$  and the sign of its eigenvalues. Convexity corresponds to the case of all positive eigenvalues, and captures the idea that risk is *ceteris paribus* preferred to uncertainty, while concavity corresponds to the case of all negative eigenvalues and captures the opposite idea.

### Example: Objective Uncertainty

Figure 1 below presents several examples of indifference maps on pure lotteries which can be obtained within our class of preferences for different choices of  $U$ .

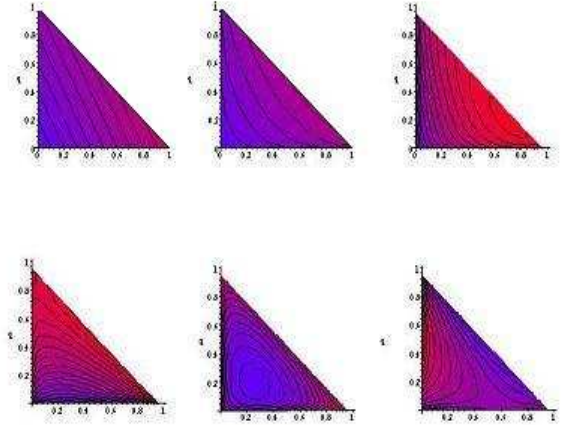


Figure 1: Examples of indifference maps on the probability triangle

The first pattern (parallel straight lines) characterizes von Neumann - Morgenstern expected utility. Within our class of representations, it corresponds to the special case of a diagonal payoff matrix  $U$ . All other patterns are impossible within von Neumann - Morgenstern expected utility.

The representation is sufficiently general to accommodate Allais' paradox from section 3. In the context of the example in section 3, let  $\{s_1, s_2, s_3\}$  be the states in which 4000, 3000, and 0 dollars are won, respectively. To accommodate

Allais' paradox, assume a slight aversion to the risk of obtaining no gain ( $s_3$ ):

$$U = \begin{array}{cc} & \begin{matrix} s_1 & s_2 & s_3 \end{matrix} \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \end{matrix} & \begin{pmatrix} 13 & 0 & -1 \\ 0 & 10 & -1 \\ -1 & -1 & 0 \end{pmatrix} \end{array}$$

Let the four lotteries  $A, B, C, D$  be defined, respectively, as the following unit vectors in  $S_+$ :  $a = (\sqrt{0.2}, 0, \sqrt{0.8})$ ;  $b = (0, \sqrt{0.25}, \sqrt{0.75})$ ;  $c = (\sqrt{0.8}, 0, \sqrt{0.2})$ ;  $d = (0, 1, 0)$ . Then lottery  $A$  is preferred to  $B$ , while  $D$  is preferred to  $C$ , as

$$\begin{aligned} u(a) &= a'Ua = 1.8, \\ u(b) &= b'Ub = 1.634, \\ u(c) &= c'Uc = 9.6, \\ u(d) &= d'Ud = 10. \end{aligned}$$

### Example: Subjective Uncertainty

In the Ellsberg puzzle, suppose that either all the non-red balls are green ( $R = 100, G = 200, B = 0$ ), or they are all blue ( $R = 100, G = 0, B = 200$ ), with equal subjective probability. Further, suppose that there are just two payoff-relevant outcomes, Win and Lose. Then the following specification of the payoff matrix accommodates the paradox.

$$U = \begin{array}{cc} & \begin{matrix} \text{Win} & \text{Lose} \end{matrix} \\ \begin{matrix} \text{Win} \\ \text{Lose} \end{matrix} & \begin{pmatrix} 1 & \alpha \\ \alpha & 0 \end{pmatrix} \end{array}$$

As we shall see, if  $\alpha = 0$  we are in the expected utility case, where the decision-maker is indifferent between risk and uncertainty; when  $\alpha > 0$ , risk is preferred to uncertainty; and when  $\alpha < 0$ , the decision-maker prefers uncertainty to risk.

In fact, let  $\{Urn1, Urn2\}$  be the set of possible states of nature, with uniform subjective probability, and let

$$\begin{aligned} r &= (\sqrt{1/3}, \sqrt{2/3}), \\ \bar{r} &= (\sqrt{2/3}, \sqrt{1/3}) \end{aligned}$$

be the lotteries representing  $R$  and  $\bar{R}$ , respectively. Furthermore, let

$$\begin{aligned} w &= (1, 0), \\ l &= (0, 1) \end{aligned}$$

be the lotteries corresponding to a sure win and a sure loss, respectively.

The mixed acts  $G$  and  $\bar{G}$  have projective expected utilities given by

$$\begin{aligned} u(G) &= p(Urn1)u(\bar{r}) + p(Urn2)u(l), \\ u(\bar{G}) &= p(Urn1)u(r) + p(Urn2)u(w). \end{aligned}$$

One also has that

$$\begin{aligned} u(w) &= 1, \\ u(l) &= 0, \\ u(r) &= r'Ur = 1/3 + \alpha\sqrt{8}/3, \\ u(\bar{r}) &= \bar{r}'U\bar{r} = 2/3 + \alpha\sqrt{8}/3, \end{aligned}$$

and therefore

$$\begin{aligned} u(g) &= 1/3 + \alpha\sqrt{2}/3 \\ u(\bar{g}) &= 2/3 + \alpha\sqrt{2}/3. \end{aligned}$$

It follows that, whenever  $\alpha > 0$ ,  $R$  is preferred to  $G$  and  $\bar{R}$  to  $\bar{G}$ , so the paradox is accommodated. When  $\alpha < 0$ , the opposite pattern emerges:  $G$  is preferred to  $R$  and  $\bar{G}$  to  $\bar{R}$ . Finally, when  $\alpha = 0$  the decision-maker is indifferent between  $R$  and  $G$ , and between  $\bar{R}$  and  $\bar{G}$ .

### Multi-Agent Decisions and Equilibrium

Within the class of preferences characterized by Theorem 4, is it still true that every finite game has a Nash equilibrium? If the payoff matrix  $U$  is diagonal we are in the classical case, so we know that any finite game has an equilibrium, which moreover only involves objective risk (in our terms, this type of equilibrium should be referred to as "pure", as it involves no subjective uncertainty). For the general case, consider that  $u(f)$  is still continuous and linear with respect to the subjective beliefs  $\pi$ , while possibly nonlinear with respect to risk. Therefore, all the necessary steps in Nash's proof (showing that the best response correspondence is non-empty, convex-valued and upper-hemicontinuous in order to apply Kakutani's fixed-point theorem) also follow in our case. Hence, any finite game has an equilibrium even within this larger class of preferences, although the equilibrium may not be pure (in our sense): in general, an equilibrium will rest on a combination of objective randomization and subjective uncertainty about other players' decisions.

### Conclusions

We presented a projective generalization of expected utility, and showed that it can accommodate the dominant decision-theoretic paradoxes. Whereas other generalizations of expected utility are typically non-linear in probabilities, projective expected utility is possibly nonlinear with respect to risk, but linear with respect to subject-related uncertainty. We found that within this class of preferences the dominant paradoxes can be accommodated, and every finite game still has an equilibrium. Moreover, the projective calculus associated with the representation should make it generally quite tractable in applications.

Our generalization of expected utility is closest to the one in Gyntelberg and Hansen (2004), which is obtained in a Savage context by postulating a non-classical (that is, non-Boolean) structure for the relevant events. By contrast, our representation is obtained in an Anscombe-Aumann context, and does not impose specific requirements on the nature of the relevant uncertainty. In particular, in our context the dominant paradoxes can be resolved even if the relevant uncertainty is of completely classical nature. Furthermore, in case the event space is non-classical, our result also provides foundations for decisions involving quantum uncertainty.

## References

Allais, M. 1953. Le Comportement de l'Homme Rationnel devant le Risque: Critique des postulats et axiomes de l'École Americaine. *Econometrica* 21: 503-546.

Bernoulli, D. 1738. Specimen theoriae novae de mensura sortis. In *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, 5: 175-192. Translated as "Exposition of a new theory of the measurement of risk". 1954, *Econometrica* 22: 123-136.

Ellsberg, D. 1961. Risk, Ambiguity and the Savage Axioms. *Quarterly Journal of Economics* 75: 643-669.

Gyntelberg, J. and Hansen, F. 2004. Expected utility theory with small worlds, Department of Economics Discussion Paper 04-20, University of Copenhagen. <http://ideas.repec.org/p/kud/kuiedp/0420.html>

von Neumann, J. 1932. *Mathematische Grundlagen der Quantenmechanik*. Berlin, Springer. Translated as *Mathematical foundations of quantum mechanics*, 1955, Princeton University Press, Princeton NJ.

von Neumann, J., and Morgenstern, O. 1944. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton NJ.

Rabin, M. and Thaler, R. H. 2001. Anomalies: Risk Aversion. *Journal of Economic Perspectives* 15: 219-232.

Savage, L. J. 1954. *The Foundations of Statistics*. Wiley, New York, NY.